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APPLICATION OF THE METHODS OF NONLINEAR PROGRAMING TO THE SOLUTION OF VARIATIONAL GAS DYNAMICS PROBLEMS

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V. G. BUTOV, I. M. VASENIN and A. I. SHELUKHA
(Tomsk)

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The method of solving variational problems of gas dynamics stated here is based on reducing them to the problems of nonlinear programing. The basic components of the computational algorithm consists of direct computation of the gas flow field and of a method of obtaining an extremum for a function of many variables. The features of the method are discussed in the course of solving the variational problems of constructing supersonic contours for maximum thrust nozzles, and in the problem of constructing the contour of a nozzle with a plane transition surface.

At present, the most commonly used method of investigating the variational problems of two-dimensional gas dynamics is the general method of Lagrange multipliers in the form first introduced in [1, 2]. Since the dependence of the optimal solution on the system of necessary extremal conditions obtained by this method is, as a rule, implicit, it means that iterative procedures must be employed to obtain it in numerical form. A step in such a procedure presumes the computation of the field of flow for the given boundaries of the region, the computation of the Lagrange multipliers over this field taking into account the equations of flow in the functional of the variational problem, and after this, a more accurate determination of the form of the contour. Considerable difficulties arise during such procedures, connected with the necessity of solving a system of partial differential equations in order to obtain the Lagrange multipliers. It is basically for this reason, that all variational problems solved up to now refer only to the supersonic flows for which the partial differential equations mentioned above are of hyperbolic type.

1. Let x and y be the rectangular coordinates. We consider an arbitrary stationary plane or axisymmetric gas dynamic flow. We require to construct a contour $y = \zeta(x)$ of an aerodynamic body immersed in this flow and producing an extremal value to the functional

$$J = \int_A^B \Phi(x, \zeta, \zeta', u_1, \ldots, u_n)\, dx \tag{1.1}$$

where Φ is a known function, $\{u_i\}$ $(i = 1, \ldots, n)$ is a system of functions satisfying the equations of flow, A and B denote the initial and the final points. Here and below the prime denotes the derivatives with respect to x taken along the contour. We consider the following isoperimetric conditions:

$$K_j = \int_A^B G_j(x, \zeta, \zeta')\, dx, \quad j = 1, \ldots, m \tag{1.2}$$

where $G_j(x, \zeta, \zeta')$ and K_j are known functions and constants. We use the form (1.1) to describe, e.g. the axial component of the nozzle thrust, the wave drag, etc.

Let us define the required optimal contour as follows:

$$y'(x) = \zeta_0(x) + \Delta\zeta(x), \quad y(x_0) = y_0 \tag{1.3}$$

where $\zeta_0(x)$ is a known function, x_0 is the initial point of the contour and $\Delta\zeta(x)$ is approximated by a segment of the series

$$\Delta\zeta(x) = \sum_{k=0}^{l} c_k\varphi_k(x) \tag{1.4}$$

Here $\{\varphi_k\}$ denotes a system of linearly independent basis functions and c_k are the coefficients. Then, for the contour specified by (1.3) and (1.4), we have

$$J = J(c_1, \ldots, c_r) \quad r<l$$

where r is chosen so that $l - r$ coefficients in (1.4) satisfy the isoperimetric conditions (1.2).

Thus, the variational problem of determining the optimal form of the contour under the specified conditions, reduces to that of determining the point $(c_1, \ldots, c_r)$ at which the function assumes an extremal value. The extremum of the function of many variables is obtained using the methods of nonlinear programing. The components of the gradient of the function J are obtained by the formulas

$$\partial J/\partial c_k \approx [J(c_1, \ldots, c_k + \Delta c_k, \ldots, c_r) - J(c_1, \ldots, c_r)] / \Delta c_k \tag{1.5}$$

$$\partial J/\partial c_k \approx [J(c_1, \ldots, c_k + \Delta c_k, \ldots, c_r) - J(c_1, \ldots, c_k - \Delta c_k, \ldots, c_r)] / (2\Delta c_k) \tag{1.6}$$

in which the function J is computed by (1.1), after computing the field of flow for the contour specified in the form (1.3) and (1.4).

If the optimal configuration of the contour must contain internal corner points the positions of which are not known in advance, then each smooth segment is approximated in the manner similar to that described in (1.3) and (1.4). In this case the coefficients of the approximating expressions describing the smooth segments with the end conditions taken into account, serve as the arguments of the function J

2. Problems arising in connection of the choice of the basis functions $\{\varphi_k\}$ and of the method of determining the extremum, were investigated on a variational problem of constructing the supersonic part of a maximum thrust nozzle for the case of an irrotational flow of a perfect gas. Various type polynomials were used as systems of the basis functions. The Chebyshev polynomials gave the best results. The example which follows shows, why these polynomials are the most suitable for use with the methods of nonlinear programing. Let

$$y' = 0.2 + \sum_{k=2}^{3} c_k\left(\frac{x}{2}\right)^k, \quad y(0) = 1 \tag{2.1}$$

$$y' = 0.2 + \sum_{k=2}^{3} c_k T_k\left(\frac{x}{2}\right), \quad y(0) = 1 \tag{2.2}$$

be two approximations to the contour of the supersonic section of an axisymmetric nozzle. Power polynomials are used in (2.1), and the Chebyshev polynomials $T_k(x/2)$ and

$0 \leqslant x \leqslant 2$ are used in (2.2). Figure 1 depicts the level lines of the function

$$J(c_2, c_3) = \int_A^B (p - p_+) yy' \, dx$$

The latter function determines, with the accuracy of up to the constant multiplier, the thrust of the supersonic sections of the nozzles. The solid lines correspond to the contours (2.1), and the dashed lines to (2.2) (the curves 1, 2 and 3 correspond to the values 0.55, 0.5, and 0.4). Here p denotes the pressure at the nozzle contour, and p_+ is the external pressure. The thrust was calculated for an irrotational axisymmetric perfect gas flow with the plane transition surface passing through the point $x = 0$.

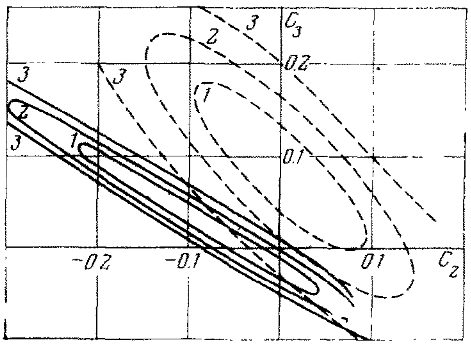

Fig. 1

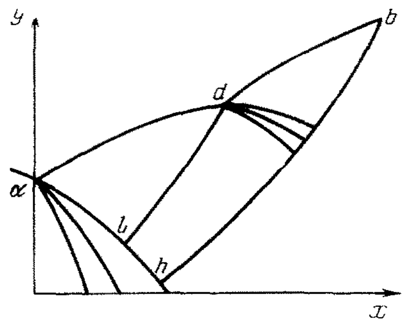

Fig. 2

The success of the application of the methods for locating an extremum of a function of many variables, which uses the approximate expressions (1.5) and (1.6) to calculate the projections of the gradient, depends to a large extent on the shape of the region in which the search is carried out. When the regions are narrow and have abrupt turns, the errors in the approximation to the gradient components calculated according to (1.5) and (1.6) are considerable, and this upsets the convergence of these methods [3]. This, together with the procedure for solving the variational problems given below, confirms the suitability of the choice of the Chebyshev polynomials as the basis functions.

A large number of computations had to be performed in order to choose a suitable method for locating the extremum of a function of many variables. Thus, in solving the variational problems of constructing axisymmetric nozzles of maximum thrust for a perfect gas we tested the Newton's method [4], the method of steepest descent and the modified Davidon method [5]. The method of [5] was found to be the most effective. Below we compare the results of the computation of the thrust produced by the supersonic section of an axisymmetric nozzle for an irrotational flow of a perfect gas, obtained by the method of steepest descent and by the method of [5], depending on the number of times the flow field was computed.

The gradient method	n	9	22	38	54	65	75	121	160
	$J \cdot 10^4$	753	764	771	780	788	790	0.0791	0.0792
The method of [5]	n	9	24	37	52	62			
	$J \cdot 10^4$	753	790	798	802	802			

The optimal nozzle contour was sought in the form

$$y' = 0.12 + \sum_{k=1}^{3} c_k T_k \left(\frac{x}{3}\right)$$
$$y(0) = 1, \quad 0 \leqslant x \leqslant 3$$

for a flow with a plane transition surface passing through the point $x = 0$, the adiabatic index equal to $\gamma = 1.14$ and the ratio $p_+ / p_0 = 0.25$ where p_0 is the stagnation pressure. The flow field was computed using the method of characteristics. The integral laws of conservation of mass and momentum were satisfied with the accuracy of up to 0. 05%. The contour obtained by the method of [5] after $n = 52$ recomputations of the flow field, and the optimal contour obtained using the general method of Lagrange multipliers, practically coincide. So do the values of the thrust functional $J = 0.0802$. Use of the method of [5] on a computer gives a solution of the problem in about 20 min. The results quoted show that the method of steepest descent converges slowly near the maximum, and this makes it unsuitable for solving the variational problems of gas dynamics.

We note that the formula (1. 5) requires fewer recomputations of the flow field than the approximation (1. 6). However, the requirement that the components of the gradient are calculated with use of the formula (1. 5) (which is a first order approximation) to a prescribed degree of accuracy, can be satisfied only by appreciable reduction of the step ΔC_k. This, in turn, leads to errors in computing the gradient components caused by the fact that the values of $J(c_1, \ldots, c_r)$ are computed approximately. In other words, the values $J(c_1, \ldots, c_k + \Delta c, \ldots, c_r)$ and $J(c_1, \ldots, c_r)$ deviate at small Δc_k by the amount comparable to the computational error in the value of the function $J(c_1, \ldots, c_r)$.

3. The supersonic part of the axisymmetric contour of a compound nozzle intended to operate in two basically different modes, is an example of an optimal configuration containing internal corner points. The complete nozzle works in the conditions of reduced external pressure p^+. Under the conditions of increased external pressure p^{+0} the final section of the complete nozzle is retracted (or jettisoned). We specify the maximum permissible length of the complete nozzle, the counterpressures p^+ and p^{+0} and the probabilities n and $1 - n$ of utilization of the complete nozzle and of its part. The optimization of the axisymmetric compound nozzle (Fig. 2) is carried out for the average thrust of the supersonic part of the nozzle which is equal, with the accuracy of up to a nonessential multiplier is

$$\chi_\Sigma = \int_a^d pyy'dx + n\int_d^b pyy'dx - \frac{n}{2} y_b^2 p^+ - \frac{1-n}{2} y_d^2 p^{+0}$$

where ad denotes the segment of the truncated nozzle and db is the final part of the nozzle. The corresponding variational problem was formulated and solved numerically using the general method of Lagrange multipliers, in [6].

To reduce this problem to the problem of nonlinear programing, we describe the contour to be determined, in the form

$$y'(x) = \begin{cases} 0.2 + \sum_{k=0}^{2} a_k T_k \left(\frac{x}{X}\right), & 0 \leqslant x \leqslant x_d \\ 0.3 + \sum_{k=0}^{2} c_k T_k \left(\frac{x - x_d}{X}\right), & x_d \leqslant x \leqslant X \end{cases}$$

where x_d is the abscissa of the point d. The ordinate of d is chosen from the condition of the contact between the initial and final section of the contour. The extremum of the thrust functional is searched for in the space of variables $\{a_0, a_1, a_2, x_d, c_0, c_1, c_2\}$.

The computations were performed to find the values of the coordinates and tangents of the angles of inclination at the characteristic points of the contour of the optimal compound nozzle, for the case of an irrotational flow of a perfect gas with the adiabatic index $\gamma = 1.4$, for the permissible length $X = 2.91$ of the nozzle and for the following values of the parameters: $p^{+0} = 0.0048$, $p^{+} = 0.116$, $n = 0.35$. (The nozzle length is referred to the radius of critical cross section, and the external pressures to the expression $\rho_* \omega_*^2$, where ω_* and ρ_* denote the critical velocity of the flow). The results obtained agree with the results of [6] where the general method of Lagrange multipliers are used.

It should be noted that the authors of [6] have solved a simpler "inverse problem" in which the values of y_a' of the corner angle of the contour at the point d $\Delta y' = y'(x_d + 0) - y'(x_d - 0)$ and of the coordinates x_l and x_h determining the positions of the points l and h on the closing characteristic of the first fan of the expansion wave (Fig. 2), were all assumed specified.

4. We illustrate the scope of application of the methods of nonlinear programing in solving the variational problems of the subsonic and transonic flows, by constructing the subsonic part of a nozzle guaranteeing a parallel sonic flow through the smallest cross section. The problem is usually solved as the inverse problem of the Laval nozzle [7] theory. However it can also be formulated as a variational problem.

Let us e. g. require to construct a contour of the subsonic part of an axisymmetric nozzle, passing through a specified point $(x_0, y_0) = (0,2)$ at the initial cross section with zero inclination and realizing, at the specified minimum cross section $x = 2$, $0 \leqslant y \leqslant 1$, a flow with a plane transition surface. To solve this problem, we approximate the required contour with the ploynomial

$$y' = -1.5x + 0.75x^2 + \sum_{k=0}^{6} c_k T_k\left(x - 1\right), \quad 0 \leqslant x \leqslant 2$$

The coefficients c_k are found from the conditions of the contour end points

$$y(0) = 2, \quad y'(0) = 0, \quad y(2) = 1, \quad y'(2) = 0$$

and from the condition of the minimum of the functional

$$J = \int_0^1 [(u - a)^2 + v^2]\, dy$$

where u and v are the projections of the velocity of the gas flow on the x- and y-axes, respectively, and a denotes the local speed of sound computed at the minimum cross section.

The flow field was calculated by the time-dependence method (*) described in [8]. The dimension of the difference grid in the xy-plane was 45×20. The integral laws of conservation of mass, momentum, and energy were satisfied with an accuracy within 0.5%.

Contour 1 obtained in the first approximation and 2 constructed as the result of solving the problem, and also projections of gas velocity u calculated at nozzle throats bounded

(*) Editors's Note. Verbatim translation from Russian is "the method of establishment".

by these contours are shown in Fig. 3. Solution of the problem had necessitated 78 reversions to the calculation of the flow field and about 20 hours time on the computer. To reduce the time required by the time-dependence method for determining gradient components, flow parameters corresponding to the contour of preceding approximation were selected as the initial field.

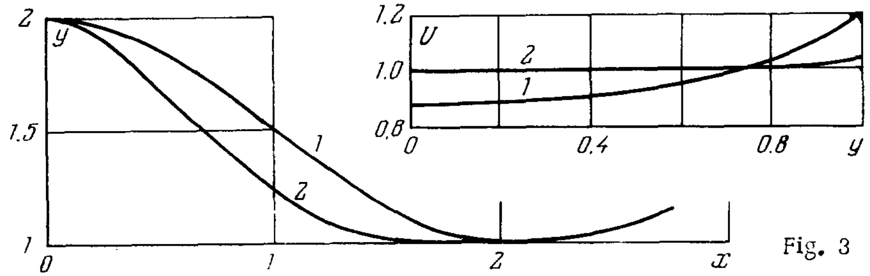

Fig. 3

In conclusion we note that the method will succeed only if the singularities of the optimal solution are known. This is illustrated by the problem of an optimal compound nozzle. In order to discover in which class of functions the solution should be sought, it is necessary to carry out a preliminary analysis of the variational problem using, e. g. the general method of Lagrange multipliers.

The main features of the proposed computational algorithm, are the calculations of the flow field, and the search for the extremum of the function of many variables using the well developed methods of nonlinear programing. This determines the chief merits of the method, namely, the simplicity of execution and the universal applicability to the variational problem of subsonic and supersonic equilibrium and nonequilibrium flows, for which methods of computing the flow fields are available.

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